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Dynamo properties of the turbulent velocity field of a saturated dynamo

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In order better to understand how dynamo systems saturate, we study the kinematic dynamo properties of velocity fields that arise from nonlinearly saturated dynamos. The technique is implemented by solving concurrently, in addition to the momentum equation, two induction equations, one for the actual magnetic field and one for an independent passive vector field. We apply this technique to two illustrative examples: convectively driven turbulence and turbulence represented by a shell model. In all cases we find that the velocity remains an efficient kinematic dynamo even after nonlinear saturation occurs. We discuss the implications to the process of dynamo saturation.

1. Introduction

The magnetization of a turbulent electrically conducting fluid is often conceptualized as a two-step process. Initially, a weak seed field is amplified by the turbulent motions. During this kinematic phase, the field is assumed to be so weak as to have no dynamical effects on the turbulence. The fluid velocity is determined solely by the external, non-magnetic forces and, from the point of view of the magnetic field evolution, it can be considered as prescribed. If, in this phase, the turbulent amplitude is stationary the average behaviour of the magnetic field is either one of exponential growth or one of exponential decay (Vainshtein & Kichatinov 1986). However, it is now commonly believed that provided the magnetic Reynolds number, which is the non-dimensional measure of advection to diffusion, is high enough the magnetic field will grow (Kazantsev 1968; Vainshtein & Kichatinov 1986; Boldyrev & Cattaneo 2004; Schekochihin *et al* 2004). With the exponential growth of the magnetic field there will be a corresponding exponential growth of the magnetic forces, which will eventually become comparable with those driving the turbulence. In this second nonlinear phase the exponential growth of the magnetic field will become saturated, and the magnetoturbulence will settle down to some stationary, well-defined level of magnetization.

Clearly in this nonlinear phase the velocity field has been modified from its original kinematic state. The question then arises as to the nature of this modification. Is there an easily identifiable property of the velocity field that in the kinematic phase leads to the exponential growth of the field whilst in the saturated state yields a statistically stationary magnetic field? Discussions of dynamo saturation can be

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divided into three broad paradigms. The first is the equipartition argument that the magnetic energy will grow until it is comparable with the kinetic energy. This is a nice criterion, as it can be applied with great generality, without knowing much about the details of the dynamo system; however it suffers from two main problems. It is easy to construct examples in which the saturation magnetic energy greatly exceeds the kinetic energy in the kinematic regime (see e.g. Stellmach & Hansen 2004) and in which it is substantially lower (Brummell, Cattaneo & Tobias 2001). Even in systems in which the energies are comparable, they may not be comparable at all scales (Vainshtein & Cattaneo 1992). Furthermore this criterion gives the level of saturation – but it does not give any hint of the mechanism for saturation. The second paradigm is a marginal stability argument that the nonlinear effects of the magnetic field are to bring the system back to marginality. This argument may be applicable to those cases in which the magnetic Reynolds number, defined by $Rm = U\ell/\eta$, where η is the magnetic diffusivity and U and ℓ are a characteristic velocity and length scale, is close to the critical value for dynamo action. However in many other cases this argument is simply not supported by the available evidence. Many astrophysical dynamos can easily be greatly supercritical, in the sense that Rm can exceed the critical value for dynamo action by many orders of magnitude. This is the case that we wish to consider in this paper, rather than the case in which Rm is close to critical. Since the magnetic diffusivity is fixed, to relax the average state to a marginal one would entail either a reduction of the velocity amplitude by several orders of magnitude or a reduction in characteristic scales by several orders of magnitude or a mixture of both. Such dramatic changes are simply not observed. To be fair, in some cases, when dynamo saturation occurs the average velocity does decrease to some extent, however, not by the huge amount required to bring Rm close to its critical value. The third paradigm is more sophisticated and invokes some subtle modification of the Lagrangian properties of the flow. It is well known that at high Rm dynamo action requires the underlying flow to be chaotic over a substantial fraction of the fluid volume (Vishik 1989; Klapper & Young 1995). Saturation could occur by suppressing the level of chaos or greatly reducing the fraction of the volume over which the flow is chaotic. In this scenario the saturated velocity differs from the kinematic velocity insofar as it has lost its chaotic properties. Indeed there are cases in which this reduction has been observed (Cattaneo, Hughes & Kim 1996). In this scenario both the chaotic stretching and dissipation become negligible, but it does not entertain the possibility that they both remain large but balance each other.

In the present paper we show that none of these arguments is satisfactory and that the mechanism for saturation is extremely elusive. There may not be a property of the velocity that can be simply calculated that would reveal whether the velocity is in the kinematic stage or the nonlinear saturated state. We study a related problem that can be precisely formulated, and while providing some useful insight into the saturation process it is much simpler to analyse. We consider the turbulent velocity associated with a saturated dynamo and ask to what extent this turbulent velocity acts as a kinematic dynamo. More specifically, we compare the turbulent velocity driven by large-scale forcing before and after the nonlinear saturation. We examine the ability of the resulting velocity to amplify at an exponential rate a passive vector field that is not necessarily everywhere aligned with the actual magnetic field but whose evolution is determined by an induction equation with the same magnetic Reynolds number as that for the magnetic field. Clearly, since both the passive field and the magnetic field obey the same equation, if they are proportional to each other at one instant they will remain proportional forever. If, on the other hand, the two fields are not everywhere aligned the passive field and magnetic field will have different evolutions. What we wish to compare are their respective growth rates. In the kinematic regime if the magnetic field grows exponentially, any passive field will eventually grow at the same rate. In this regime, one can regard the dynamo growth rate as an average property of the velocity, since it does not depend on which vector field it is applied to. It is not immediately obvious whether this will continue to hold in the nonlinear regime. By definition, in the saturated state the growth rate of the magnetic field is zero. But is this true for other vector fields, satisfying the induction equation? If the answer is yes, then the saturation property is an average property in the same sense as above. If the answer is no, then the saturation property (by which we mean the property of neither growing nor decaying on average) is specific to some restricted class of vector fields, one of which is the actual magnetic field, and does not hold in general. In this case the magnetic field will saturate, but the velocity field will remain chaotic.

We apply this technique to two illustrative examples: convectively driven turbulence and magnetohydrodynamic (MHD) turbulence represented by a shell model. Convective turbulence is a natural choice; it is known to be an effective dynamo; its properties can be carefully controlled; and it can be efficiently represented numerically. Shell models, on the other hand, provide an idealization of a turbulent flow in which all degrees of freedom at a given wavenumber are represented by a single (complex) coefficient. They share some of the properties of the full systems, but lack their geometrical complexity.

2. Convective dynamos: formulation

We consider dynamo action driven by (Boussinesq) convection in a rotating plane layer. Using standard notations the evolution equations can be written as

$$(\partial_t - \sigma\nabla^2)\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \sigma Ta^{1/2}\boldsymbol{e}_z \times \boldsymbol{u} = -\nabla p + \boldsymbol{J}\times\boldsymbol{B} + \sigma Ra\,\theta\boldsymbol{e}_z\ , \tag{2.1}$$

$$(\partial_t - \sigma/\sigma_m\nabla^2)\boldsymbol{B} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}\ , \tag{2.2}$$

$$(\partial_t - \nabla^2)\theta + \boldsymbol{u}\cdot\nabla\theta = w\ , \tag{2.3}$$

$$\nabla\cdot\boldsymbol{B} = \nabla\cdot\boldsymbol{u} = 0\ , \tag{2.4}$$

where $\boldsymbol{J} = \nabla\times\boldsymbol{B}$ is the current density; w is the vertical velocity; and θ denotes the temperature fluctuations relative to a linear background profile (e.g. Chandrasekhar 1961). In this non-dimensionalization magnetic fields are measured in units of the Alfvén velocity. Four dimensionless numbers appear explicitly: the Rayleigh number Ra, the Taylor number Ta and the kinetic and magnetic Prandtl numbers σ and σ_m.

In the horizontal directions we assume that all fields are periodic with periodicity λ. In the vertical we consider standard illustrative boundary conditions for the temperature, velocity and magnetic field, namely

$$\theta = w = \partial_z u = \partial_z v = B_z = \partial_z B_x = \partial_z B_y = 0 \quad \text{at } z = 0, 1. \tag{2.5}$$

We supplement (2.1)–(2.4) by an extra equation for the evolution of a (solenoidal) passive field $\boldsymbol{Z}$ that obeys the same induction equation (2.2) as $\boldsymbol{B}$,

$$(\partial_t - \sigma/\sigma_m\nabla^2)\boldsymbol{Z} + \boldsymbol{u}\cdot\nabla\boldsymbol{Z} = \boldsymbol{Z}\cdot\nabla\boldsymbol{u}, \quad \nabla\cdot\boldsymbol{Z} = 0, \tag{2.6}$$

together with the same boundary conditions. We stress again here that $\boldsymbol{Z}$ is passive and does not act back on $\boldsymbol{u}$. This implies that whilst $\boldsymbol{u}$ and $\boldsymbol{B}$ satisfy an energy equation (with a coupling term $\langle\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla\boldsymbol{u})\rangle$ between the equations) there is no corresponding

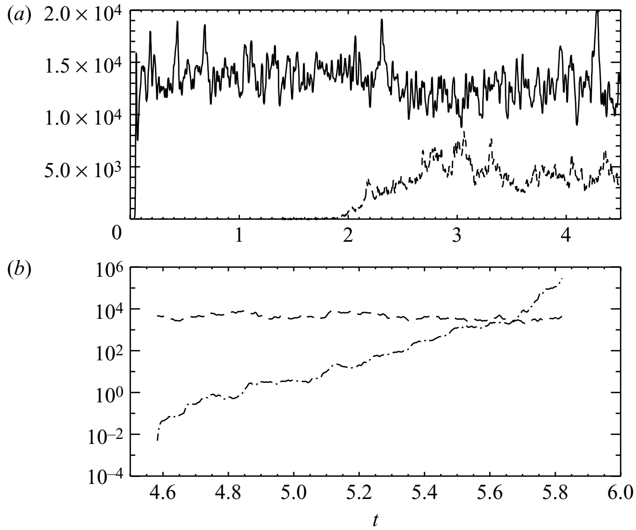

FIGURE 1. Convective dynamo time series. (*a*) Time series for the kinetic energy (solid line) and 5× magnetic energy (dashed line) showing the dynamo evolution to a saturated state. (*b*) Time series for the magnetic energy (dashed line) and the energy of the passive field (dot-dashed line); note that this is logarithmic to show the exponential growth of the passive field energy.

coupling between $\boldsymbol{u}$ and $\boldsymbol{Z}$. Hence, as stated above, if $\boldsymbol{B}$ and $\boldsymbol{Z}$ are not initially aligned they will have different evolutions. However the root mean square (r.m.s.) value of the $\boldsymbol{Z}$ field could be stationary, decay or grow, and it it these dynamics we wish to ascertain.

We solve (2.1)–(2.6) numerically by standard pseudo-spectral methods (see, for example, Cattaneo, Emonet & Weiss 2003). For recent publications on convectively driven dynamos we note, for example, the works of Stellmach & Hansen (2004) and Cattaneo & Hughes (2006).

3. Convective dynamos: results

The convective model equations (2.1)–(2.6) were integrated for a wide range of parameters; with varying degrees of supercriticality (as measured by Ra) and rotation rates (as measured by Ta). For all cases, whether rotating or non-rotating, the results were qualitatively similar, and so we focus here on describing the results for one representative case, with $Ta = 0$ and $Ra = 100\,000$. The other parameters are set at $\sigma = 1.0$, $\sigma_m = 5.0$ and $\lambda = 3.0$. Note that setting $\sigma_m = 5.0$ makes the dynamo easy to excite and has the tendency to generate magnetic fields on smaller length scales than typical velocity scales.

Figure 1(*a*) shows the typical evolution of the convective dynamo system. We first integrate the purely hydrodynamic system until a statistically steady convecting state, which consists of a number of moderately turbulent convective eddies interacting nonlinearly, is achieved. Once this hydrodynamic steady state is established (by $t = 1$) a seed magnetic field $\boldsymbol{B}_s$ is introduced – in this stage the passive vector field $\boldsymbol{Z}$ remains zero. There follows a typical dynamo evolution for a convective flow. The seed magnetic field is amplified exponentially on an advective time scale (with a growth rate $\sigma_{\boldsymbol{B}} \approx 5.9$), and the system rapidly saturates in a turbulent MHD state. Figure 1(*a*) shows that the magnetic energy of the saturated state reaches approximately 7 % of

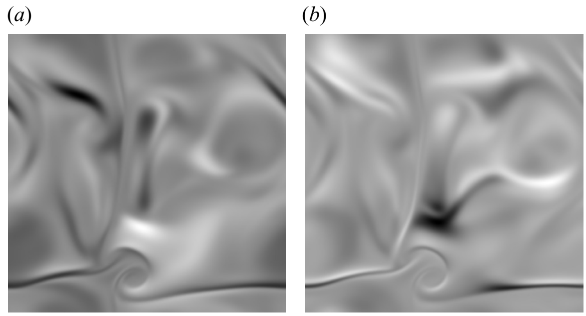

FIGURE 2. Density plots showing typical form of (*a*) the magnetic field and (*b*) the passive field for the convective model. Shown are the x components of both fields at the level $z \approx 0.1$.

the saturated kinetic energy, which is itself reduced slightly (of the order of 10 %) from its kinematic value. It is the dynamo properties of this saturated velocity field that is of interest.

To this end, we continue solving the equations for $\boldsymbol{u}$ and $\boldsymbol{B}$, integrating the nonlinear saturated state forward in time. At $t \approx 4.6$ we introduce a random seed passive vector field $\boldsymbol{Z}_s$ into the linear equation (2.6). Figure 1(*b*) shows the evolution of the saturated magnetic energy and the energy of the passive vector field as the calculation is then continued. As expected the saturated magnetic energy remains statistically steady throughout the evolution, but it is clear that the passive vector field is exponentially amplified by the saturated velocity field; the saturated velocity field *does indeed* act perfectly well as a dynamo! The growth rate here is $\sigma_{\boldsymbol{Z}} \approx 5.3 < \sigma_{\boldsymbol{B}}$. In the set of calculations we have performed for these convective dynamos, $\sigma_{\boldsymbol{Z}}$ is always less than $\sigma_{\boldsymbol{B}}$, sometimes by as much as 50 %. The ratio appears to depend on the parameters of the system but not significantly on the initial conditions. We stress here that if the initial passive vector field is not chosen to be random but aligned with the saturated magnetic field – i.e. $\boldsymbol{Z}_s = C\boldsymbol{B}$, with C a constant – then $\boldsymbol{Z} = C\boldsymbol{B}$ for all times in the subsequent evolution.

Figure 2 compares the typical spatial form of the saturated magnetic field $\boldsymbol{B}$ with that of the exponentially growing passive field $\boldsymbol{Z}$ in the form of density plots. It is clear that both fields have very similar spatial structures – both are concentrated on small scales. This is perhaps not surprising, as they are advected by the same velocity field at high magnetic Reynolds number, yet the passive field is growing exponentially, whilst the magnetic field does not grow or decay on average. We stress again that these results are typical and arise whether the system is moderately or highly turbulent – or rotating or non-rotating.

4. Shell model dynamos: formulation

The results for the turbulent convective system described above seem counter-intuitive, and there is the possibility that they may be model specific. We investigate this possibility by considering the simplest possible models of hydromagnetic dynamo action, namely shell models.

In the context of hydrodynamic turbulence, shell models have long been constructed with the aim of reproducing the spectral properties of turbulence within a low-order model (see e.g. Gledzer 1973; Yamada & Ohkitani 1987). More recently these shell models have been extended to include the effects of magnetic fields, to examine both dynamo action and MHD turbulence (see e.g. Frick 1983; Plunian & Stepanov 2007). These models are constructed specifically to reproduce certain conservation laws that are inherent in the dissipation-free fluid and full MHD systems and therefore to conserve ideal invariants.

Here we consider the dynamics of a relatively simple MHD shell model proposed by Frick & Sokoloff (1998; hereinafter FS98), in order to discuss the kinematic dynamo properties of a turbulent saturated dynamo. This local shell model conserves ideal invariants and can be shown to reproduce some dynamics of two-dimensional and three-dimensional MHD turbulence and dynamo action. As in the previous sections, we consider the evolution of the velocity field ($\boldsymbol{u}$), magnetic field ($\boldsymbol{B}$) and the passive vector field ($\boldsymbol{Z}$). Following FS98 we consider the dynamics on a range of spatial wavenumbers $k_n = k_0\lambda^n$, $0 \leqslant n \leqslant n_{max}$ and consider the complex variable $U_n(t)$ as representative of all the modes of $\boldsymbol{u}$ in the shell with a wavenumber k such that $k_n \leqslant k \leqslant k_{n+1}$. Similar representations for the magnetic fields and passive vector field are given by the complex coefficients $B_n(t)$ and $Z_n(t)$.

The basic shell model as introduced in FS98 describes the evolution of the velocity coefficients (U_n) and the magnetic field coefficients (B_n) via the system

$$\dot{U}_n + Re^{-1}k_n^2 U_n = \mathrm{i}k_n \left\{ (U^*_{n+1}U^*_{n+2} - B^*_{n+1}B^*_{n+2}) - \frac{\epsilon}{2}(U^*_{n-1}U^*_{n+1} - B^*_{n-1}B^*_{n+1}) - \frac{(1-\epsilon)}{4}(U^*_{n-2}U^*_{n-1} - B^*_{n-2}B^*_{n-1}) \right\} + f_n, \tag{4.1}$$

$$\dot{B}_n + Rm^{-1}k_n^2 B_n = \mathrm{i}k_n \left\{ (1-\epsilon-\epsilon_m)(U^*_{n+1}B^*_{n+2} - B^*_{n+1}U^*_{n+2}) + \frac{\epsilon_m}{2}(U^*_{n-1}B^*_{n+1} - B^*_{n-1}U^*_{n+1}) + \frac{(1-\epsilon_m)}{4} \times (U^*_{n-2}B^*_{n-1} - B^*_{n-2}U^*_{n-1}) \right\}, \tag{4.2}$$

where $*$ represents the complex conjugate; Re and Rm are the non-dimensional fluids and magnetic Reynolds numbers; and f_n is a random forcing acting on only a few shells near $n=0$. Here ϵ and ϵ_m are parameters, which are set at $\epsilon = 1/2$, $\epsilon_m = 1/3$ in order to conserve the relevant invariants (i.e. the total energy, the cross-helicity and the magnetic helicity) for non-dissipative three-dimensional dynamics (as in FS98). The only other parameter of the model is the spacing of the shells in wavenumber space (λ), which we set to $\lambda = (\sqrt{5}+1)/2$, which is the minimum spacing allowed and is believed to lead to the most accurate results (see Plunian & Stepanov 2007).

This system of equations is able to describe regular dynamo action, and if the variables B_n are set to zero, then the system reduces to the hydrodynamic Gledzer–Ohkitani–Yamada (GOY) model. The dynamics of this system is described in detail in FS98. Here, as in the last two sections, we focus on achieving a saturated dynamo, where U_n and B_n reach statistically steady states, and examine the dynamo properties of the saturated velocity field U_n. This is achieved by simultaneously solving (4.1) and (4.2) together with the evolution equation for Z_n, which, of course, is identical to (4.2). It is the dynamics of this system of equations that will be investigated in the next section.

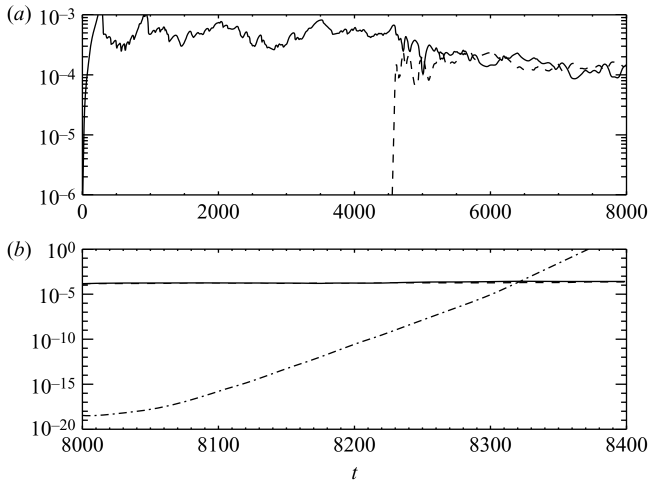

FIGURE 3. Shell model dynamo time series. (*a*) Time series for the kinetic energy (solid line) and magnetic energy (dashed line) showing the dynamo evolution to a saturated state. (*b*) Time series for the kinetic energy (solid line), magnetic energy (dashed line) and the energy of the passive field (dot-dashed line).

5. Shell model dynamos: results

In this section, we repeat the procedure of § 3 for the shell model equations. Once again we integrate these equations for a wide variety of parameters and obtain qualitatively similar results each time. We present here a typical evolution, with $n_{max} = 19$, corresponding to $k_{max} \approx 9349$. We fix $Re = Rm = 10^6$ (so the magnetic Prandtl number is unity) and choose $f_n = 10^{-4}(1 + i)\,\delta_{n6}$ (so that steady forcing is applied at $n = 6$). This choice of steady forcing leads to the driving of flows with a non-zero helicity; in shell models the helicity is defined as $H_u = \sum_n 0.5(-1)^n k_n |U_n|^2$.

The hydrodynamic equations are integrated until a statistically steady state is achieved, as shown in figure 3(*a*). As for the convective model, this state has a complicated temporal evolution about a well-defined mean. Once the hydrodynamic solution has settled down, a small seed magnetic field is introduced and the dynamo evolution followed. Once again the seed magnetic field grows exponentially before saturating in a statistically steady MHD state (as shown in figure 3*a*). In this case the dynamo is very efficient, and the magnetic energy saturates in equipartition with the kinetic energy. Figure 4 shows the spectra for the velocity and magnetic fields in the saturated state; it is clear from this that the shell model has saturated the magnetic field so that it is in equipartition with the velocity field at each scale in the nonlinear regime. Interestingly the magnetic field does have significant power at small k in the saturated state for this choice of parameters, having been localized to larger k in the kinematic regime.

With the magnetic field and velocity field in a statistically steady saturated state, a weak passive seed field is added by setting $Z_n \neq 0$. The evolution of this passive field is compared with that of the saturated velocity and magnetic fields in figure 3(*b*). Once again the passive field grows exponentially, somewhat surprisingly with a growth rate larger than the kinematic growth rate for the magnetic field. Figure 4 also compares the spectrum for the exponentially growing passive field at a representative time with that for the saturated magnetic field. It is clear that, in this case, the growing

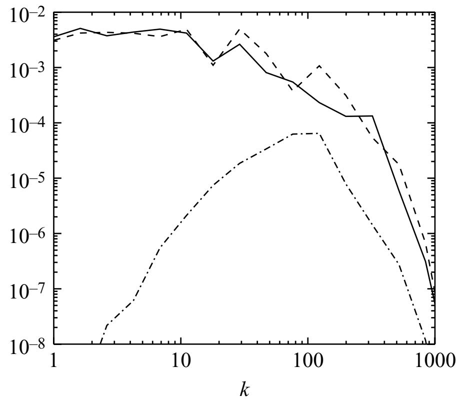

FIGURE 4. Spectra for the saturated velocity field (solid line), saturated magnetic field (dashed line) and kinematic passive field (dot-dashed line).

passive solution is more localized at high k than the saturated magnetic field. We stress again that although the details of the solutions are parameter dependent (with the relative growth rate of the passive field to that for the magnetic field being a sensitive function of parameters) the exponential amplification of the passive vector field remains a robust feature of the dynamics. However the precise dynamics of the variables Z_n are a function of the chaotic nature of the attractor in (U_n, B_n) space: we find cases in which the variables Z_n have growth rates less than and greater than B_n – and these will be presented in a subsequent paper.

6. Discussion

In this paper we have addressed the issue of how dynamos saturate and have argued that this process is very subtle and not in concord with any of the previously suggested theories. In particular we have shown that in the saturated state the velocity remains a good kinematic dynamo for all passive vector fields that are not everywhere aligned with the magnetic field. Remarkably this holds both for full MHD systems (here we have analysed the specific case of convection) and for shell models. This implies that the dynamo does not saturate either by relaxing the system to a state close to marginality or by suppressing the chaotic stretching in the flow. Furthermore, because this result applies equally to full MHD and shell models, it suggests that the dynamo saturation relies on temporal rather than spatial correlations.

The origin of these correlations clearly comes from the Lorentz force term in the momentum equation. This causes a special relationship (in the form of correlations) between the velocity and the magnetic fields that is not shared by any other vector field that only satisifies the induction equation. It is therefore apparent that the mechanism by which saturation occurs cannot be captured by analysing the induction equation alone.

This result also has some implications for the generation of large-scale fields. Thus far we have discussed the kinematic dynamo properties of a saturated velocity, with no particular distinction between large- and small-scale dynamo action. However we could focus this question on the case in which the saturated velocity is helical, or more generally lacks reflexional symmetry, and address the issue of large-scale dynamo action. What would be the evolution of a large-scale passive field advected by the saturated helical velocity? There are only two possibilities. Either the passive field is everywhere aligned with the actual magnetic field, in which case it will neither grow nor decay, or it is not aligned with the actual field, in which case it will quickly latch onto the fastest growing 'eigenfunction' and grow at the same rate as any other perturbation, small or large (Boldyrev & Cattaneo 2004; Cattaneo & Hughes 2008). In general this fastest growing eigenfunction may be dominated by the large or small scales, but for the cases we considered the small scales dominated.

Finally we would like to note that any argument that attempts to describe the saturation process (for dynamos of any scale) via the induction equation alone, for example arguments that rely solely on the evolution of magnetic helicity, are likely to be misleading. In our examples $\boldsymbol{B}$ and $\boldsymbol{Z}$ both satisfy the same equations for the evolution of the integrated magnetic helicity, yet their behaviour is very different – one grows exponentially, whilst the other is statistically steady. As noted above, any argument for the level and mechanism of saturation must take into account the correct solution of the momentum equation together with any constraints that arise from the induction equation.

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